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BUCKLING OF THIN CONICAL SHELLS UNDER UNIFORM EXTERNAL PRESSURE

TECHNICAL REPORT NO. WAL TR 836.32/3

BY
JOHN F. MESCALL

FEBRUARY 1961

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WATERTOWN ARSENAL
WATERTOWN 72, MASS.

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TITLE

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UNDER UNIFORM EXTERNAL PRESSURE

ABSTRACT

The stability of thin circular conical shells under uniform external pressure is investigated by means of an energy procedure. Appropriate strain displacement relations are formulated, paying particular attention to the influence of rotation effects. Higher ordered terms are retained in the strain energy expression and a buckling criterion is established on the basis of the vanishing of the second variation of the total energy functional. Numerical estimates of the critical pressure were made for a wide range of shell geometries by employing a Rayleigh-Ritz procedure.

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INTRODUCTION

The problem we wish to investigate is that of the elastic stability of thin circular conical shells under uniform external pressure. Both conical frustra and complete cones are to be considered.

In those cases of shell stability where it is difficult or impossible to solve the problem exactly, the direct methods of the variational calculus deserve special attention. The more complex the problem, the more advantageous is the use of such methods.

Several theoretical investigations of the problem of conical shell stability employing an energy procedure have been carried out in recent years without resolving the problem. One of the underlying problems of stability analyses, and the probable source of much of the disagreement, is that they are beset with considerable algebraic and numerical difficulties which necessitate the introduction of approximations. None of these approximate theories yield results in close agreement with one another or with experiment over a wide range of geometries.

In the interest of simplicity most investigators simplify the problem by neglecting, in the strain energy expression, those displacement terms of cubic and higher order, and employing the principle of virtual displacements. This states, "Of all the displacements satisfying given boundary conditions, those which also satisfy the equilibrium conditions make the total potential energy, π, assume a stationary value." In variational terms, this corresponds to setting the first variation of the total potential energy equal to zero.

However, such a procedure minimizes the influence of rotation terms on the deformation. In thin flexible bodies such as shells, this implies an effective stiffening of the shell and a higher estimate of the critical load.

The object of this investigation is to present a systematic procedure to account more adequately for the influence of rotation effects in thin shells. We first introduce a form of strain displacement relations in which specific emphasis is placed on the rotation terms. Next, higher-ordered terms are retained in the expression for the total potential energy of the system, and an appropriate stability criterion is formulated.

Now, if a functional such as π is quadratic in its independent variables and their derivatives, a stationary value is unique provided it exists. (The Euler equations and the natural boundary conditions are linear.) If the functional is of higher order in its independent variables, multiple stationary values may exist. (The Euler equations are nonlinear.) This introduces the possibility of alternative equilibrium positions some of which are stable, others not. An equilibrium position is designated as as stable when the virtual work is negative for all admissible "virtual displacements," i.e., when additional external forces are necessary to displace the system from its equilibrium position. In variational notation, an extremal defining a stable equilibrium position must maximize the work functional or minimize the total potential energy functional, π. Since (for small virtual displacements) an extremal is a minimum when the second variation is positive definite (i.e., positive for all admissible argument functions in the neighborhood of the extremal), the criterion for stable equilibrium is $\delta_2\pi > 0$. By the same token, an unstable equilibrium position is characterized by $\delta_2\pi < 0$. Hence the condition for the onset of instability for thin shells is $\delta_2\pi = 0$.[1]

The theory thus formulated is valid for small virtual displacements about the equilibrium position, and is sometimes referred to as the infinitesimal theory of buckling. It was first proposed by Trefftz[2] and was employed in the study of the buckling of cylindrical shells under external pressure by Langhaar and Boresi.[3] and more recently, in the analysis of the buckling of a cylinder under circumferential band loads by Brush and Field.[4]

The procedure outlined, with the general strain-displacements given in Appendix A, is, then generally applicable to the stability of thin shells. Explicit estimates of critical loadings may be made with the aid of the Rayleigh-Ritz method. It is shown in the body of the report, however, that in using this method care must be taken to select a set of displacement functions which are realistic over the entire range of geometries to be considered.

DERIVATION OF POTENTIAL ENERGY OF THE SYSTEM

We consider the problem of the elastic stability of a conical shell to be one in the nonlinear theory of elasticity, with small strains and small but finite deflections.

An appropriate formulation of the strain-displacement relations for such a problem is derived in Appendix A in terms of the thin shell coordinate system (α_1, α_2, z), where α_1, α_2 are lines of principal curvature on the shell's middle surface, having principal radii of curvature R_1 and R_2, and Lamé coefficients H_1, H_2 of the form

$$H_1 = A_1(1 + z/R_1)$$

$$H_2 = A_2(1 + z/R_2) \ ,$$

and z is the thickness coordinate, normal to (α_1, α_2). In terms of the middle surface displacements (u, v, w) in the (α_1, α_2, z) directions, the strain displacement relations may be written as:

$$
\left.
\begin{aligned}
\epsilon_{11} &= e_{11} + \tfrac{1}{2}\left(\omega_2^2 + \omega_3^2\right) + z\left\{k_{11} - \frac{\omega_2 k_{13}}{2} + \omega_3\left[\frac{k_{12} - k_{21}}{2}\right]\right\} \\
\epsilon_{22} &= e_{22} + \tfrac{1}{2}\left(\omega_1^2 + \omega_3^2\right) + z\left\{k_{22} + \frac{\omega_1 k_{23}}{2} + \omega_3\left[\frac{k_{12} - k_{21}}{2}\right]\right\} \\
\epsilon_{12} &= e_{12} + e_{21} - \omega_1\omega_2 + z\left\{k_{12} + k_{21} + \frac{\omega_1 k_{13}}{2} - \frac{\omega_2 k_{23}}{2}\right\}
\end{aligned}
\right\}
\quad (1)
$$

where:

$$
\left.
\begin{aligned}
e_{11} &= \frac{1}{A_1}\frac{\partial u}{\partial\alpha_1} + \frac{v}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{w}{R_1}
& e_{21} &= \frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} - \frac{v}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \\
e_{12} &= \frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{u\partial A_1}{A_1A_2\partial\alpha_2}
& e_{22} &= \frac{1}{A_2}\frac{\partial v}{\partial\alpha_2} + \frac{u}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \frac{w}{R_2} \\
e_{13} &= \frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}
& e_{23} &= \frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2} \\
2\omega_1 &= -\psi + \frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}
& 2\omega_2 &= \phi - \frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} + \frac{u}{R_1} \\
2\omega_3 &= \frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{u}{A_1A_1}\frac{\partial A_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} + \frac{v}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \\
k_{11} &= \frac{1}{A_1}\frac{\partial\phi}{\partial\alpha_1} + \frac{\psi}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \chi/R_1
& k_{21} &= \frac{1}{A_2}\frac{\partial\phi}{\partial\alpha_2} - \frac{\psi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \\
k_{12} &= \frac{1}{A_1}\frac{\partial\psi}{\partial\alpha_1} - \frac{\phi}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}
& k_{22} &= \frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{\phi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \chi/R_2 \\
k_{13} &= \frac{1}{A_1}\frac{\partial\chi}{\partial\alpha_1} - \phi/R_1
& k_{23} &= \frac{1}{A_2}\,\partial\chi/\partial\alpha_2 - \psi/R_2
\end{aligned}
\right\}
(2)
$$

and, finally,

$$\phi = - e_{13} \qquad \psi = - e_{23} \qquad \chi = e_{11} + e_{22} \tag{3}$$

We may further simplify these relations and still retain the same degree of approximation already inherent in them by observing that ω_3, the component of rotation about the z axis, will be very small compared to the remaining components of rotation ω_1 and ω_2. This is so because the shell is very stiff ("massive") in its own plane. Accordingly we may omit the nonlinear terms containing ω_3 in the above expressions for the strains.

We may specialize to conical geometry with the aid of Figures 1A and 1B. For conical coordinates:

$$\left.\begin{array}{lll} \alpha_1 = \xi & A_1 = 1 & \dfrac{1}{R_1} = o \\ \\ \alpha_2 = \theta & A_2 = r = R_2 \sin\alpha = R_o - \xi\cos\alpha & \end{array}\right\} \tag{4}$$

and the appropriate nonlinear strain-displacement relations are:

$$\left.\begin{aligned}
\epsilon_\xi &= \epsilon_{11} = \frac{\partial u}{\partial \xi} + \tfrac{1}{2}\left(\frac{\partial w}{\partial \xi}\right)^2 \\
\epsilon_\theta &= \epsilon_{22} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} - u\cos\alpha + w\sin\alpha\right) + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta} - v\sin\alpha\right)^2 \\
\gamma_{\xi\theta} &= \epsilon_{12} = \frac{\partial v}{\partial \xi} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + v\cos\alpha\right) + \frac{1}{r}\frac{\partial w}{\partial \xi}\left(\frac{\partial w}{\partial \theta} - v\sin\alpha\right) \\
K_\xi &= K_{11} = -\frac{\partial^2 w}{\partial \xi^2} \\
K_\theta &= K_{22} = -\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \sin\alpha\frac{\partial v}{\partial \theta}\right) + \frac{\cos\alpha}{r}\frac{\partial w}{\partial \xi} + \frac{\sin\alpha}{r}\frac{\partial u}{\partial \xi} \\
&\quad + \frac{\sin\alpha}{r^2}\left(\frac{\partial v}{\partial \theta} - u\cos\alpha + w\sin\alpha\right) \\
K_{\xi\theta} &= K_{12} = -\frac{1}{r}\left(2\frac{\partial^2 w}{\partial \xi\partial \theta} - \sin\alpha\frac{\partial v}{\partial \xi}\right) - \frac{2\cos\alpha}{r^2}\left(\frac{\partial w}{\partial \theta} - v\sin\alpha\right).
\end{aligned}\right\} \tag{5}$$

It should be pointed out that the derivation of equations 1 involves no order of magnitude estimates on the relative values of the displacements themselves, but only on the over-all character of the deformation. For this reason it is believed they are more generally applicable to the stability problem of shells. For example, in formulating Donnell's equation, the number of circumferential waves in the buckling pattern is assumed to be large. On this basis, w is assumed to be small compared with $\frac{\partial^2 w}{\partial\theta^2}$, and v is assumed negligible compared with $\frac{\partial w}{\partial\theta}$.

The strain energy of a conical shell is given by

$$U = U_m + U_b \tag{6}$$

where

$$\left.\begin{aligned} U_m &= \frac{Eh}{2(1-\nu^2)} \int_0^{2\pi} \int_0^1 \left(\epsilon_\xi^{\,2} + \epsilon_\theta^{\,2} + 2\nu\, \epsilon_\xi \epsilon_\theta + \left(\frac{1-\nu}{2}\right)\gamma_{\xi\theta}^{\,2}\right) r\,d\theta\,d\xi \\ U_b &= \frac{Eh^3}{24(1-\nu^2)} \int_0^{2\pi} \int_0^1 (K_\xi^{\,2} + K_\theta^{\,2} + 2\nu K_\xi K_\theta + 2(1-\nu)K_{\xi\theta}^{\,2})\, r\,d\theta\,d\xi \end{aligned}\right\} \tag{7}$$

where E = Young's modulus, ν = Poisson's ratio and h = shell thickness. Finally, the potential energy of the system is given by

$$\pi = U_m + U_b - U_{ext} \tag{8}$$

where U_{ext} is the change in potential of the external forces. In Appendix B we have shown that for an external pressure p,

$$\left.\begin{aligned} U_{ext} &= p \sin\alpha \iint (\xi + u)\{v_\xi(w_\theta - v) - (R + w)_\xi(v_\theta + w + r)\}\,d\xi\,d\theta \\ &\quad + p \sin\alpha \left\{\frac{21A_1}{3} + e(A_1 + A_2) - \frac{1}{3}\left(A_2 + \sqrt{A_1A_2}\right)\right\} \end{aligned}\right\} \tag{9}$$

where A_1 and A_2 are the cross-sectional areas at the top and bottom of the cone, and e is the average extension of the ends of the cone.

DETERMINATION OF DISPLACEMENT FUNCTIONS

The Rayleigh-Ritz procedure is a convenient technique for approximating the solution of buckling problems where exact solution is difficult. It results in an upper bound estimate for the critical load. The method involves assuming a form for displacements which satisfies the boundary conditions and which contains a set of arbitrary parameters. These parameters are then chosen to optimize the quantity desired.

Ideally, the chosen set of displacement functions should also present a reasonable description of the actual displacements occurring during the deformation of the shell. The estimate of the critical load is fairly sensitive to the choice of displacement functions.

Another way of expressing this is that an optimal set of displacement functions used in a Rayleigh-Ritz procedure is one which yields a good approximation to those displacements which actually satisfy the equilibrium equations of the shell and yet remain simple enough in form to be of practical benefit.

Niordson[5] and Radkowski[6,7] employed the (cylindrical) displacement functions

$$\left.\begin{aligned} u_1 &= A \cos n\,\theta \cos \pi\xi/\ell \\ v_1 &= B \sin n\,\theta \sin \pi\xi/\ell \\ w_1 &= C \cos n\,\theta \sin \pi\xi/\ell \end{aligned}\right\} \qquad (10)$$

which satisfy conditions of simple support at both ends of a truncated cone and are also the exact solutions for a cylindrical shell of finite length. They apparently yield satisfactory results for very short conical frustra. In terms of the important "taper ratio" parameter $\beta = (l \cos\alpha)/ro$, these functions are good estimates of the buckling pattern for β small compared with unity. Niordson limited the application of his results to $\beta < 1/3$. Radkowski retained terms neglected by Niordson and thereby extended the range of application of his results, but found that as β approached unity, these displacements introduce a singularity into the expression for p_{cr}. Radkowski used a physical argument to extrapolate his results to complete cones. Grigolyuk[8] selected the displacement functions

$$\left.\begin{aligned} u_1 &= Ar^2 \cos\left(\frac{\pi r}{ro}\right)\sin n\,\theta \\ v_1 &= Br^2 \sin\left(\frac{\pi r}{ro}\right)\cos n\,\theta \\ w_1 &= Cr^2 \sin\left(\frac{\pi r}{ro}\right)\sin n\,\theta \end{aligned}\right\} \qquad (11)$$

which satisfy the apex conditions for a complete cone:

$$u = v = w = u,_r = v,_r = w,_r = o \quad \text{at } r = o.$$

However, they are not satisfactory for a truncated cone unless modified somewhat. For example, a more satisfactory form for w_1 for truncated and complete cones might be:

$$w_1 = Cr^2 \sin \pi\left(\frac{r - ro}{r_1 - ro}\right) \sin n\,\theta .$$

The buckling displacement functions employed in the present investigations are of the form:

$$\left.\begin{aligned} u_1 &= A\left[f_1(\beta)\left(\frac{r}{ro}\right)^2 + f_2(\beta)\right]\cos\,(ar + b)\,\cos\,n\,\theta \\ v_1 &= B\left[f_1(\beta)\left(\frac{r}{ro}\right)^2 + f_2(\beta)\right]\sin\,(ar + b)\,\sin\,n\,\theta \\ w_1 &= C\left[f_1(\beta)\left(\frac{r}{ro}\right)^2 + f_2(\beta)\right]\sin\,(ar + b)\,\cos\,n\,\theta \end{aligned}\right\} \qquad (12)$$

where

$$a \equiv \frac{\pi}{r_1 - ro} \qquad b \equiv -\frac{\pi ro}{r_1 - ro}$$

and

$$f_1(\beta) \to 1, \quad f_2(\beta) \to o \quad \text{as } \beta \to 1$$

while:

$$f_1(\beta) \to o, \quad f_2(\beta) \to 1 \quad \text{as } \beta \to o.$$

It may easily be seen that this choice of displacements satisfies suitable edge conditions for both complete and truncated cones. Moreover, for suitable f_1 and f_2 it should reproduce the dependence of the buckling pattern on the slant length of the cone - viz.: for short frustra the dimpled region extends over a much larger percentage of the slant length of the cone.

In particular, Figure 2 contains a plot of $w_1/w_{1\,max.}$ as a function of the slant length, ξ/ℓ, for several values of β. In this graph we have selected

$$f_1(\beta) = \frac{2\beta}{1+\beta^2}, \quad f_2\,(\beta) = (1-\beta)^2. \tag{13}$$

Examination of Figure 2 points out why one would not expect the Niordson-Radkowski displacements ($\beta = o$) to be suitable for complete cones $\beta = 1$, and conversely, why Grigolyuk's functions ($\beta = 1$) would not be suitable as $\beta \rightarrow o$.

On the other hand, Seide[9] who has employed a power series solution of the equilibrium equations has also computed the displacement w which emerges from his solution. A comparison of Figure 2 with his results indicates very good qualitative agreement.

Our displacement functions in final form, then, are:

$$\left.\begin{aligned} u &= u_o + A\left[\frac{2\beta}{1+\beta^2}\left(\frac{r}{ro}\right)^2 + (1-\beta)^2\right]\cos\,(ar+b)\,\cos n\,\theta \\ v &= v_o + B\left[\frac{2\beta}{1+\beta^2}\left(\frac{r}{ro}\right)^2 + (1-\beta)^2\right]\sin\,(ar+b)\,\sin n\,\theta \\ w &= w_o + C\left[\frac{2\beta}{1+\beta^2}\left(\frac{r}{ro}\right)^2 + (1-\beta)^2\right]\sin\,(ar+b)\,\cos n\,\theta \end{aligned}\right\} \tag{14}$$

where we recall that u_o, v_o and w_o are to be determined from the linear (pre-buckling) theory.

If we assume that the pre-buckling state of the shell is adequately described (away from the edges) by the membrane theory of thin shells, then we may write for the direct stress resultants in the cone:

$$N_\xi = -\frac{pr}{2\sin\alpha}, \quad N_\theta = -\frac{pr}{\sin\alpha}. \tag{15}$$

From the linear strain-displacement and stress-strain laws we have

$$\epsilon_\xi = \frac{\partial u}{\partial \xi} = \frac{1}{Eh}\,(N_\xi - \nu N_\theta),$$

so:

$$u_o = \frac{(1 - 2\nu)\ pr^2}{4Eh \sin\alpha \cos\alpha} . \tag{16}$$

To find w_o we have:

$$\epsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_\xi) = \frac{1}{r}(w \sin\alpha - u \cos\alpha) ,$$

so:

$$w_o = -\frac{3pr^2}{4Eh \sin^2\alpha} . \tag{17}$$

Due to rotational symmetry prior to buckling, $v_o = o$.

DETERMINATION OF THE BUCKLING LOAD

The stability of the system can be investigated by considering a virtual displacement from a loaded but unbuckled equilibrium configuration and examining the corresponding second variation of π.

If to u_o, v_o and w_o (the pre-buckling equilibrium configuration) are added the (virtual) displacements u_1, v_1 and w_1 then π becomes:

$$\pi + \Delta\pi = \pi + \delta_1\pi + \frac{\delta_2\pi}{2!} + \frac{\delta_3\pi}{3!} + \frac{\delta_4\pi}{4!} \tag{18}$$

where $\delta_1\pi$ is linear in u_1, v_1, w_1

$\delta_2\pi$ is quadratic in these variables, etc.

Substitution of the chosen forms for the displacements into the expression for the total potential energy π and integration of the resulting definite integrals is a straightforward, if tedious procedure. The expression for $\delta_2\pi$ then has the form:

$$\delta_2\pi = a_{11}A^2 + 2a_{12}AB + 2a_{13}AC + a_{22}B^2 + 2a_{23}BC + a_{33}C^2 \tag{19}$$

where the a_{ij} will be defined momentarily.

The system will be in stable equilibrium if $\delta_2\pi > o$, unstable if $\delta_2\pi < o$. Hence the condition for the onset of instability is

$$\delta_2\pi = o, \tag{20A}$$

or, in view of the fact that $\delta_2\pi$ is a quadratic form in terms of A, B, C:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = o. \tag{20B}$$

Each of the terms a_{ij} may be written in the form

$$a_{ij} = \alpha_{ij} + \frac{p}{E}\beta_{ij} \; . \tag{21}$$

Hence the stability criterion is obtained from a solution of Equation 20B for p/E. Upon expanding this determinant into a cubic equation in p/E, it was observed that the cubic and quadratic terms could be neglected compared with the constant term and the term linear in p/E. The resulting equation, when solved for p_{cr}/E yields:

$$\frac{p_{cr}}{E} = \frac{\alpha_{11}(\alpha_{23}^2 - \alpha_{22}\alpha_{33}) + \alpha_{12}^2\alpha_{33} + \alpha_{22}\alpha_{13}^2 - 2\alpha_{12}\alpha_{13}\alpha_{23}}{\left[\alpha_{11}(\alpha_{22}\beta_{33} + \alpha_{33}\beta_{22}) - 2\alpha_{11}\alpha_{23}\beta_{23} - \alpha_{12}^2\beta_{33} + 2\alpha_{12}(\alpha_{13}\beta_{23} + \alpha_{23}\beta_{13}) - 2\alpha_{22}\alpha_{13}\beta_{13} - \beta_{22}\alpha_{13}^2\right]} \; . \tag{22}$$

In arriving at the expressions for the a_{ij}, the following simplifying approximations were made: (a) Quadratic terms in the pre-buckling displacements u_o, v_o and w_o (which are directly proportional to p/E) were neglected; (b) Fourth degree terms in u_1, v_1 and w_1 were neglected; (c) In view of the form of the chosen displacements

$$\left(\text{e.g.} \quad w_1 = C\left\{f_1(\beta)\left(\frac{r}{ro}\right)^2 \sin(ar + b)\cos n\,\theta + f_2(\beta)\sin(ar + b)\cos n\,\theta\right\}\right),$$

each a_{ij} is of the form:

$$a_{ij} = f_1^2(\beta)(a_{ijI}) + f_2^2(\beta)(a_{ijII}) + f_1(\beta)f_2(\beta)(a_{ijIII}). \quad (23)$$

Now, apart from their values as β approaches zero and unity, the forms of f_1 and f_2 were left unspecified. We now require further that f_1 and f_2 be chosen such that the cross-product terms become negligible. The amount of error introduced by this approximation can be systematically evaluated and reduced by the judicious choice of f_1 and f_2. A little experimentation reveals that the forms chosen in Equation 13 are quite satisfactory: i.e. neglecting the cross-product terms does not significantly affect the determination of the buckling load. We may, then, regard a_{ij} to be of the form:

$$a_{ij} = f_1^2(\beta)\left[\overset{*}{\alpha}_{ij} + \frac{p}{E}\overset{*}{\beta}_{ij}\right] + f_2^2(\beta)\left[\overset{**}{\alpha}_{ij} + \frac{p}{E}\overset{**}{\beta}_{ij}\right] \quad (24)$$

where the single and double asterisks indicate the contribution from the first and second portions, respectively, of the assumed displacement forms.

Omitting an unessential constant factor $Eh/2(1-\nu^2)$, the non-vanishing $\overset{*}{\alpha}_{ij}$ and β_{ij}^* are given by:

$$\left.\begin{aligned}
\overset{*}{\alpha}_{11} &= -\pi\cos\alpha\left\{I^* + \frac{h^2\sin^2\alpha}{12\ ro^2}\ II^*\right\} \\
\overset{*}{\alpha}_{12} &= n\pi\left\{III^* + \frac{h^2\sin^2\alpha}{6\ ro^2}\ IV^*\right\} \\
\overset{*}{\alpha}_{13} &= \pi\sin\alpha\left\{III^* + \frac{h^2}{12\ ro^2}\left[(n^2 + \sin^2\alpha)\ IV^* - \cos^2\alpha VI^*\right]\right\} \\
\overset{*}{\alpha}_{22} &= \frac{-\pi}{\cos\alpha}\left\{n^2\ VII^* + 2(1-\nu)\cos^2\alpha X^* + \frac{h^2\sin^2\alpha}{12\ ro^2}\left[4n^2\ XIII^*\right.\right. \\
&\qquad \left.\left. + 2(1-\nu)\cos^2\alpha XII^*\right]\right\} \\
\overset{*}{\alpha}_{23} &= -n\pi\tan\alpha\left\{VII^* + \frac{h^2}{12\ ro^2}\left[2(n^2 + \sin^2\alpha)XIII^* - 2\cos^2\alpha XIV^*\right]\right\}
\end{aligned}\right| \quad (25)$$

$$\alpha_{33}^{*} = -\pi \cos\alpha \left\{ \tan^2\alpha(\mathrm{VII}^*) + \frac{h^2}{12\,ro^2}\left[\cos^2\alpha(\mathrm{XI}^* + \mathrm{XII}^*) \right.\right.$$

$$- 2(n^2 + \sin^2\alpha)\mathrm{XVII}^* + 2\nu\cos^2\alpha\,\mathrm{XVIII}^* + 8(1-\nu)n^2\,\mathrm{XII}^*$$

$$\left.\left. + \mathrm{XIII}^*\left(\frac{(n^2+\sin^2\alpha)^2}{\cos^2\alpha} + 8(1-\nu)(n^4 - 2n^2)\right)\right]\right\}$$

$$\beta_{13}^{*} = \frac{-3\pi}{2}\cot^2\alpha\left(\frac{ro}{h}\right)\mathrm{V}^* \tag{25}$$

$$\beta_{22}^{*} = \frac{\pi\tan\alpha}{4}\left(\frac{ro}{h}\right)\left[\mathrm{VIII}^* \frac{-12(1-\nu)\mathrm{IX}^*}{\tan^2\alpha} + 8(1-\nu^2)\mathrm{XIX}^*\right]$$

$$\beta_{23}^{*} = \frac{n\pi}{4\cos\alpha}\left(\frac{ro}{h}\right)\left[\mathrm{VIII}^* - 6(1-2\nu)\cot^2\alpha\,\mathrm{IX}^* + 8(1-\nu^2)\sin\alpha\,\mathrm{XIX}^*\right]$$

$$\beta_{33}^{*} = 3\pi\cot\alpha(1-2\nu)\left(\frac{ro}{h}\right)\left[\mathrm{XV}^* + \frac{n^2}{2\cos^2\alpha}\mathrm{XVI}^* + \tan^2\alpha\left(\frac{4[1-\nu^2]}{1-2\nu}\right)\mathrm{XIX}^*\right]$$

where I* through XIX* are polynomials in β, and are tabulated in Appendix C(a).

The components α_{ij}^{**} and β_{ij}^{**} arising from the second portion of the displacement functions may be evaluated in terms of the Sinus and Cosinus integrals as follows:

$$\alpha_{11}^{**} = -\frac{\pi^3\cos\alpha}{2}\left(\frac{1}{2} - \frac{1}{\beta}\right) + \beta\cos\alpha I_2^{**} + \frac{h^2\sin^2\alpha}{12\,ro^2}\left[\beta\cos\alpha I_4^{**}\right.$$

$$\left. + \pi^2\cos\alpha I_1^{**}/\beta + \pi\cos\alpha I_7^{**}\right]$$

$$\alpha_{12}^{**} = -\frac{n}{2}\left\{\beta I_3^{**} + \nu\pi^2 + \frac{h^2\sin^2\alpha}{6\,ro^2}\left[\beta I_5^{**} + 2\pi I_8^{**}\right]\right\} \tag{26}$$

$$\alpha_{13}^{**} = -\left\{\frac{\beta\sin\alpha I_3^{**}}{2} + \frac{\nu\pi^2\sin\alpha}{2} + \frac{h^2\sin\alpha}{24\,ro^2}\left[\beta(n^2 + \sin^2\alpha)I_5^{**}\right.\right.$$

$$+ \pi^2(1+\nu)\cos^2\alpha I_3^{*}/\beta + \nu\pi^4\cos^2\alpha/\beta^2 + 2\pi I_9^{**}\cos^2\alpha$$

$$+ 2\pi I_8^{**}(n^2 + \sin^2\alpha)\Big]\Big\}$$

$$a_{22}^{**} = \Bigg\{ \frac{n^2 I_1^{**}\beta\tan\alpha}{\sin\alpha} + (1-\nu)\pi^3\cos\alpha\left(\frac{1}{\beta} - \frac{1}{2}\right) + \frac{h^2\sin^2\alpha}{12\ ro^2}$$

$$\Big[4\beta I_6^{**}(n^2 + \big[2(1-\nu)\big]\cos^2\alpha)/\cos\alpha + 2(1-\nu)\pi^2\cos\alpha I_2^{**}/\beta$$

$$+ 4\pi(1-\nu)\cos\alpha I_7^{**}\Big]\Bigg\}$$

$$a_{23}^{**} = n\Bigg\{ \beta\tan\alpha I_1^{**} + \frac{h^2\sin^2\alpha}{12\ ro^2}\Bigg[\frac{4(n^2 + \sin^2\alpha)\beta I_6^{**}}{\sin\alpha\ \cos\alpha}$$

$$+ \frac{4\nu\pi^2\cot\alpha I_1^{**}}{\beta} + \frac{8(1-\nu)\pi^2\cot\alpha I_2^{**}}{\beta} + 16(1-\nu)\beta\cot\alpha I_6^{**}$$

$$+ \pi\cot\alpha I_7^{**}(2 + 12[1-\nu]\Bigg]\Bigg\}$$

$$a_{33}^{**} = \Bigg\{ \beta\tan\alpha\sin\alpha I_1^{**} + \frac{h^2\sin^2\alpha}{12\ ro^2}\Bigg[\frac{\pi^5\cos\alpha}{2\beta^2\tan^2\alpha}\ \frac{1}{\beta} - \frac{1}{2} \qquad (26)$$

$$+ \frac{\beta(n^2 + \sin^2\alpha)^2 I_6^{**}}{\sin^2\alpha\cos\alpha} + \frac{\pi^2(\cos^2\alpha + 8n^2(1-\nu)I_2^{**}\cos\alpha}{\beta\sin^2\alpha}$$

$$+ \frac{2\nu\pi^2(n^2 + \sin^2\alpha)I_1^{**}\cos\alpha}{\beta\sin^2\alpha} + \frac{8(1-\nu)n^4\beta\cos\alpha I_6^{**}}{\sin^2\alpha}$$

$$+ \frac{\pi\cos\alpha I_7^{**}(n^2[9-8\nu] + \sin^2\alpha}{\sin^2\alpha}\Bigg]\Bigg\}$$

$$\beta_{13}^{**} = \frac{3(1+\nu)\pi^2\beta\left(\frac{1}{2} - \frac{1}{\beta}\right)(\cot^2\alpha)}{4}\left(\frac{ro}{h}\right)$$

$$\beta_{22}^{**} = -\beta\tan\alpha\left(\frac{ro}{h}\right)\Bigg\{\frac{\nu\pi(1-2\nu)}{4} + \frac{3(1-\nu)\pi\cot^2\alpha}{4} + \frac{\pi(1-\nu^2)}{2}$$

$$+ (1 - \nu/2)\left[I_8^{**} - \frac{2\beta I_{10}^{**}}{\pi} + \frac{\beta^2 I_{11}^{**}}{\pi^2}\right]\Bigg\}$$

$$\beta_{23}^{**} = - \frac{\beta n\pi\sin^2\alpha}{\cos\alpha} \left(\frac{ro}{h}\right)\left\{\frac{3(1-2\nu)\cot^2\alpha}{8} + \frac{\nu(1-2\nu)}{4} + \frac{(1-\nu^2)\sin\alpha}{2}\right.$$

$$\left. + \frac{(2-\nu)}{2\pi}\left[I_8^{**} - \frac{2\beta}{\pi} I_{10}^{**} + \frac{\beta^2}{\pi^2} I_{11}^{**}\right]\right\}$$

$$\beta_{33}^{**} = \beta\left(\frac{ro}{h}\right) (\cot\alpha)\left\{\frac{3\nu\pi}{2} - \frac{(1-2\nu)n^2\pi\nu}{2\cos^2\alpha} - \pi(1-\nu^2)\tan^2\alpha\right. \tag{26}$$

$$- \frac{n^2(2-\nu)}{\cos^2\alpha}\left[I_8^{**} - \frac{2\beta I_{10}^{**}}{\pi} + \frac{\beta^2}{\pi^2} I_{11}^{**}\right]$$

$$\left. + \pi^3\left[\left(\frac{1}{6} + \frac{1}{4\pi^2}\right)\left(\frac{\nu}{2} - 1\right) - \nu(1-2\nu)\left(\frac{1}{8} + \frac{1}{2}\left[\frac{1}{4\pi^2} - \frac{1}{12}\right]\right)\right]\right\}$$

$$+ \pi^3\cos\alpha\left(\frac{ro}{h}\right)\left(1 - \frac{1}{\beta}\right)\left(\frac{1-\nu^2}{2}\right)$$

where I_1^{**} through I_{11}^{**} are in turn given in terms of the Sinus and Cosinus Integrals, and are tabulated in Appendix C(b).

RESULTS AND DISCUSSION

Equations 22, 25 and 26 determine the value of the critical pressure. These relations were programmed for a digital computer and p_{cr}/E was determined for the following set of parameters:

$\alpha = 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 85^\circ$

$\frac{h}{\rho} = \frac{h \sin\alpha}{ro} = .02, .01, .005, .002, .001, .0005, .00025$ and $.0001$

$\beta = .05, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$

and $\nu = .3$ in all cases.

When these parameters are inserted into Equation 22, p_{cr}/E still remains a function of n_1 the number of lobes in the circumferential direction. Since we are seeking the minimum buckling load, p_{cr}/E was minimized with respect to n, subject to the restriction that n is an integer greater than one. The results of these computations are plotted in Figures 3 through 8.

Comparison of these curves with the corresponding results of Radkowski (Reference 6) reveals that the present results are, in general, lower estimates of p_{cr}/E. The difference between the results diminishes for small β, where the conical frustra behaves more like a cylinder, and where Niordson (Reference 5) and Radkowski (References 6 and 7) are essentially in agreement. The difference in results, then, reflects the β dependence of the buckled shape. Furthermore, our results verify Radkowski's (Reference 6 and 7) physical hypothesis that when plotted versus β, p_{cr}/E does not increase significantly from its minimum value (attained at or near $\beta = .6$) as β tends toward unity.

The results obtained in this report for complete cones ($\beta = 1$), when compared with the corresponding results of Grigolyuk (Reference 8) are substantially lower for larger values of h/ρ. As h/ρ decreases, the difference between the results diminishes, and for very small values of h/ρ, (very thin shells) the two solutions would apparently coincide.

A comparison of our results with those obtained by Seide (Reference 9) is complicated because of the difference in the selection of parameters and the sensitivity of the results when interpolation is used. For this reason we have computed p_{cr}/E for the same parameters employed in Reference 9 for two values of the angle: $\alpha = 85^{\circ}$ and $\alpha = 30^{\circ}$. The ratio of our results to Seide's results for these parameters is given in Table I.

Inspection reveals that the results of this report seem to be in closer agreement with the results of Reference 9 than other energy solutions. However a difference in the dependence of the solutions upon h/ρ is noted: for higher values of h/ρ our results lie below those of Reference 9 but as h/ρ becomes very small, our results become larger.

This difference in the independence of p_{cr}/E upon h/ρ might well be made the basis of a careful experimental investigation. It reflects, partially, the influence of the bending portion of the strain energy which is generally thought to have much less significance than the stretching energy.

The procedure employed in this report, then, when used with suitably chosen displacement functions, yields numerical estimates of critical pressure which tend to be lower than those obtained by applying the Rayleigh-Ritz method to the classical strain-energy procedure.

TABLE I

R = RATIO OF RESULTS OF PRESENT INVESTIGATION TO THOSE OF REFERENCE 4

$\alpha = 85^\circ$			$\alpha = 30^\circ$		
β	R	h/ρ	β	R	h/ρ
0.041911	1.032	0.0038	0.4641	0.7186	0.00107
	1.102	0.00191		0.8336	0.000535
	1.154	0.00127		0.9056	0.000357
	1.194	0.000954		0.9546	0.000268
	1.309	0.000477		1.0871	0.000134
0.08045	1.0199	0.00366	0.6340	0.8547	0.000732
	1.116	0.00183		0.9751	0.000366
	1.177	0.00122		1.045	0.000244
	1.232	0.000916		1.1045	0.000183
	1.372	0.000458		1.248	0.0000915
0.1489	0.9517	0.00339	0.7760	1.068	0.000448
	1.054	0.00169		1.212	0.000224
	1.121	0.00113		1.235	0.000149
	1.168	0.000848		1.305	0.000112
	1.317	0.000424		1.462	0.000056
0.3043	0.6787	0.00277	0.8965	1.271	0.000207
	0.7840	0.00139		1.296	0.0001045
	0.8377	0.000924		1.355	0.0000690
	0.8924	0.000693		1.411	0.00005175
	0.9899	0.000347		1.477	0.00002588
0.4666	0.6104	0.00213	0.9454	1.298	0.0001092
	0.7111	0.00106		1.355	0.0000546
	0.7765	0.000708		1.391	0.0000364
	0.8066	0.000531		1.458	0.0000273
	0.9081	0.000266		1.505	0.00001365
0.6363	0.7131	0.00145	0.9719	1.365	0.0000562
	0.8038	0.000725		1.436	0.0000281
	0.8753	0.000483		1.465	0.00001873
	0.9100	0.000362		1.505	0.00001405
	1.0291	0.000181		1.604	0.000007025

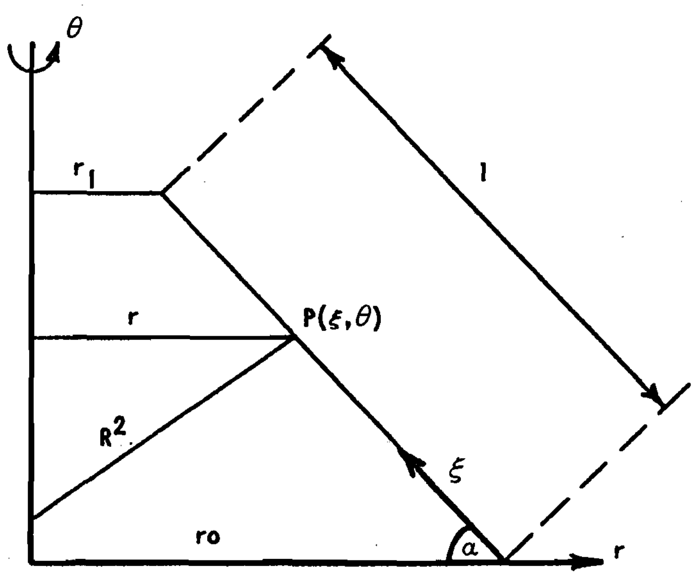


FIGURE 1A

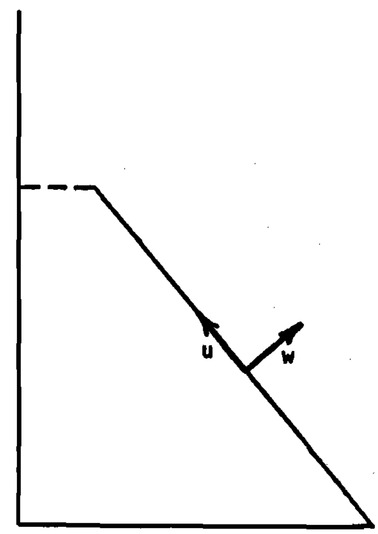


FIGURE 1B

DEFINITION OF COORDINATE SYSTEM

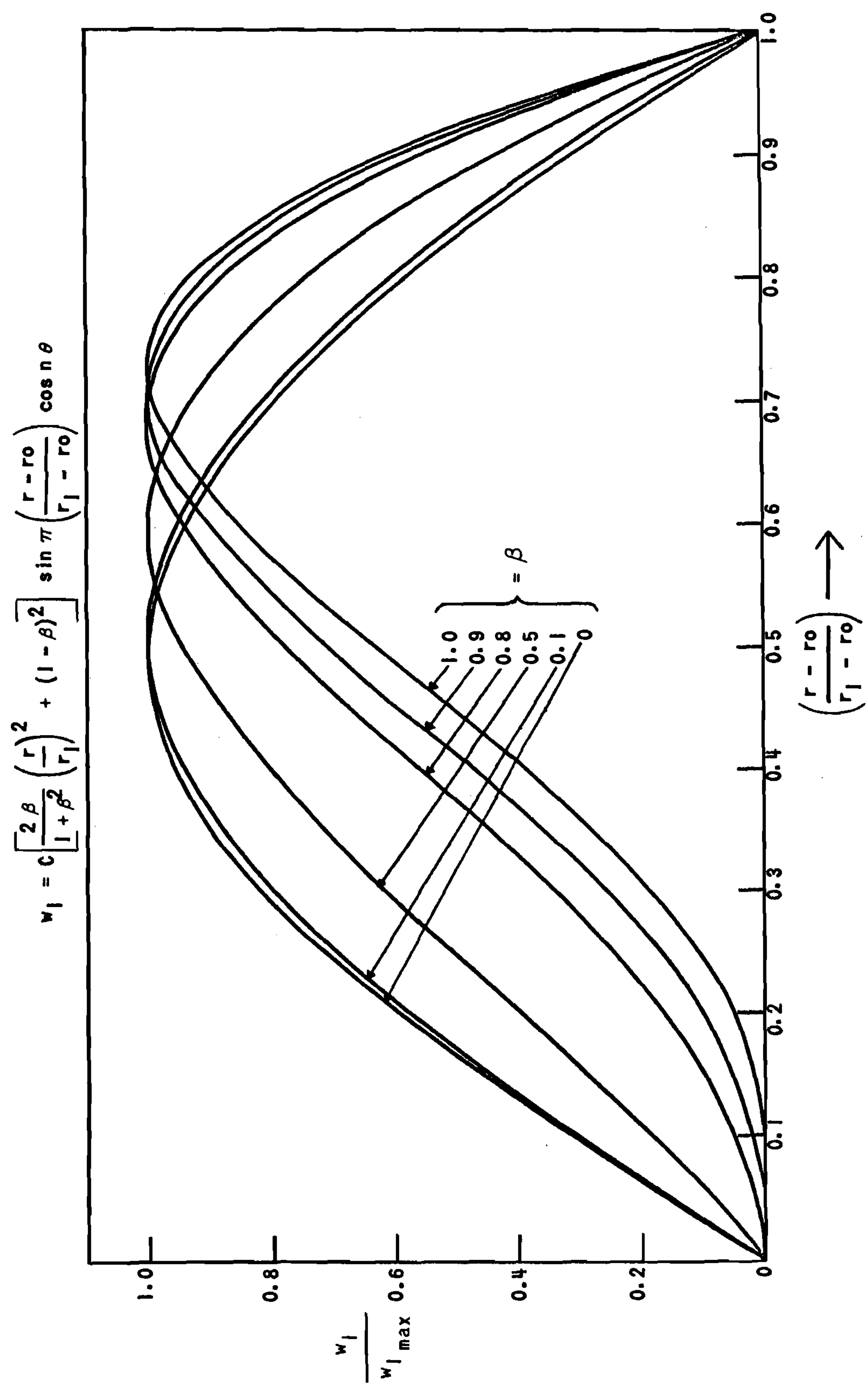
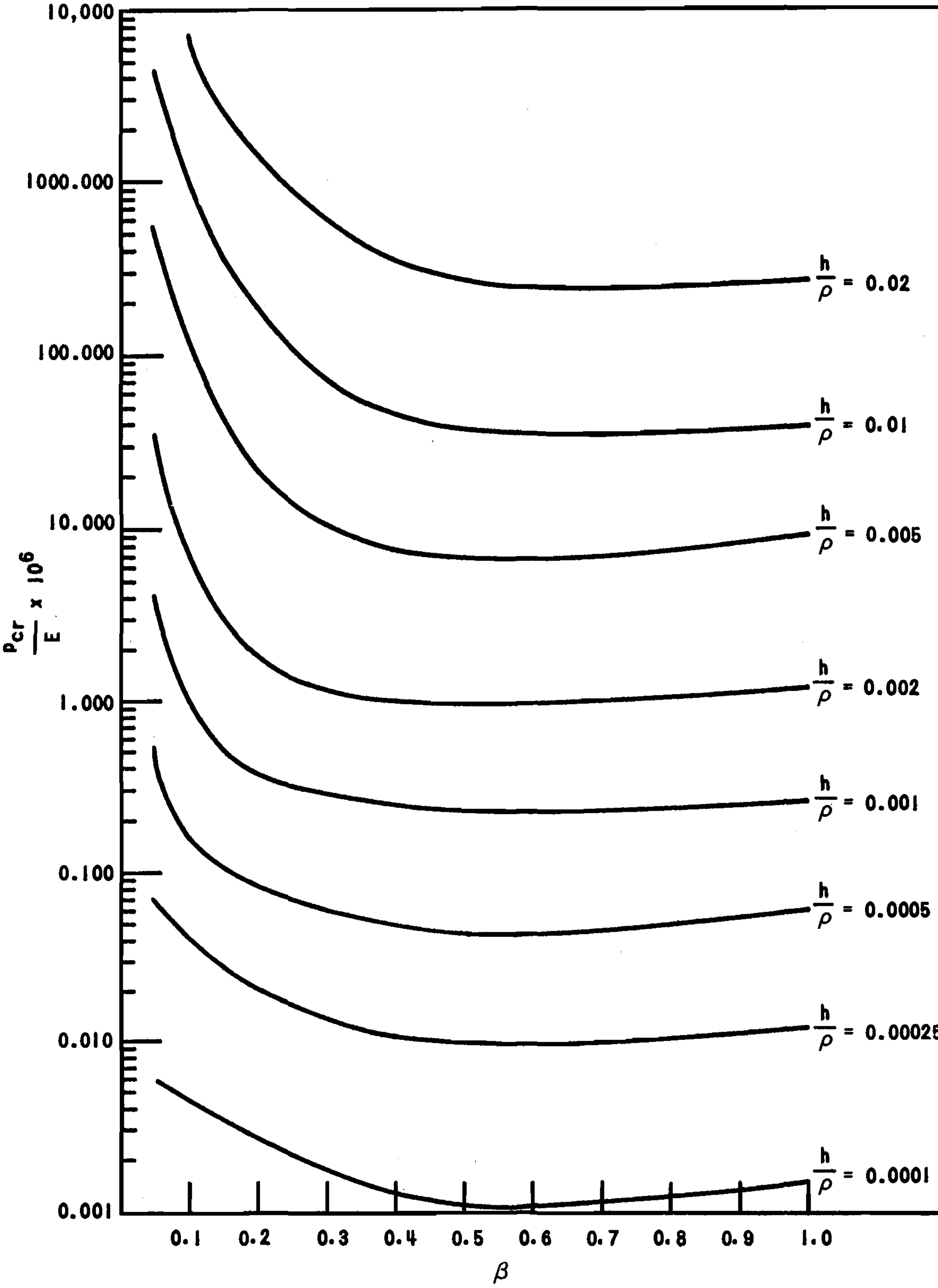

$w_1 = C\left[\frac{2\beta}{1+\beta^2}\left(\frac{r}{r_1}\right)^2 + (1-\beta)^2\right] \sin\pi\left(\frac{r-ro}{r_1-ro}\right)\cos n\theta$
1.0
0.9
0.8
0.5
0.1
0
$= \beta$
$\left(\frac{r-ro}{r_1-ro}\right) \longrightarrow$
$\frac{w_1}{w_{1\,max}}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

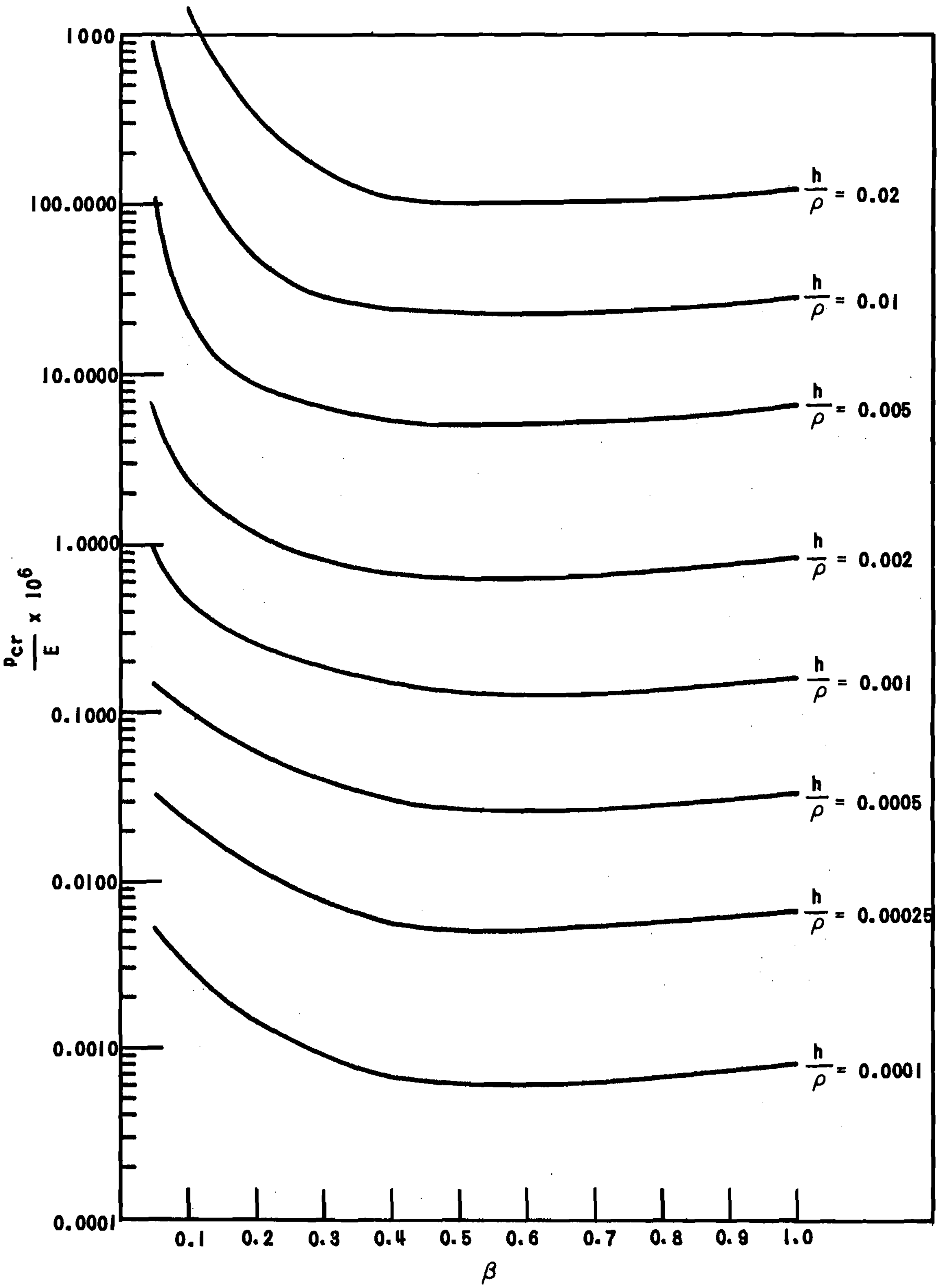
FIGURE 2



CRITICAL PRESSURE FOR CONICAL FRUSTRA

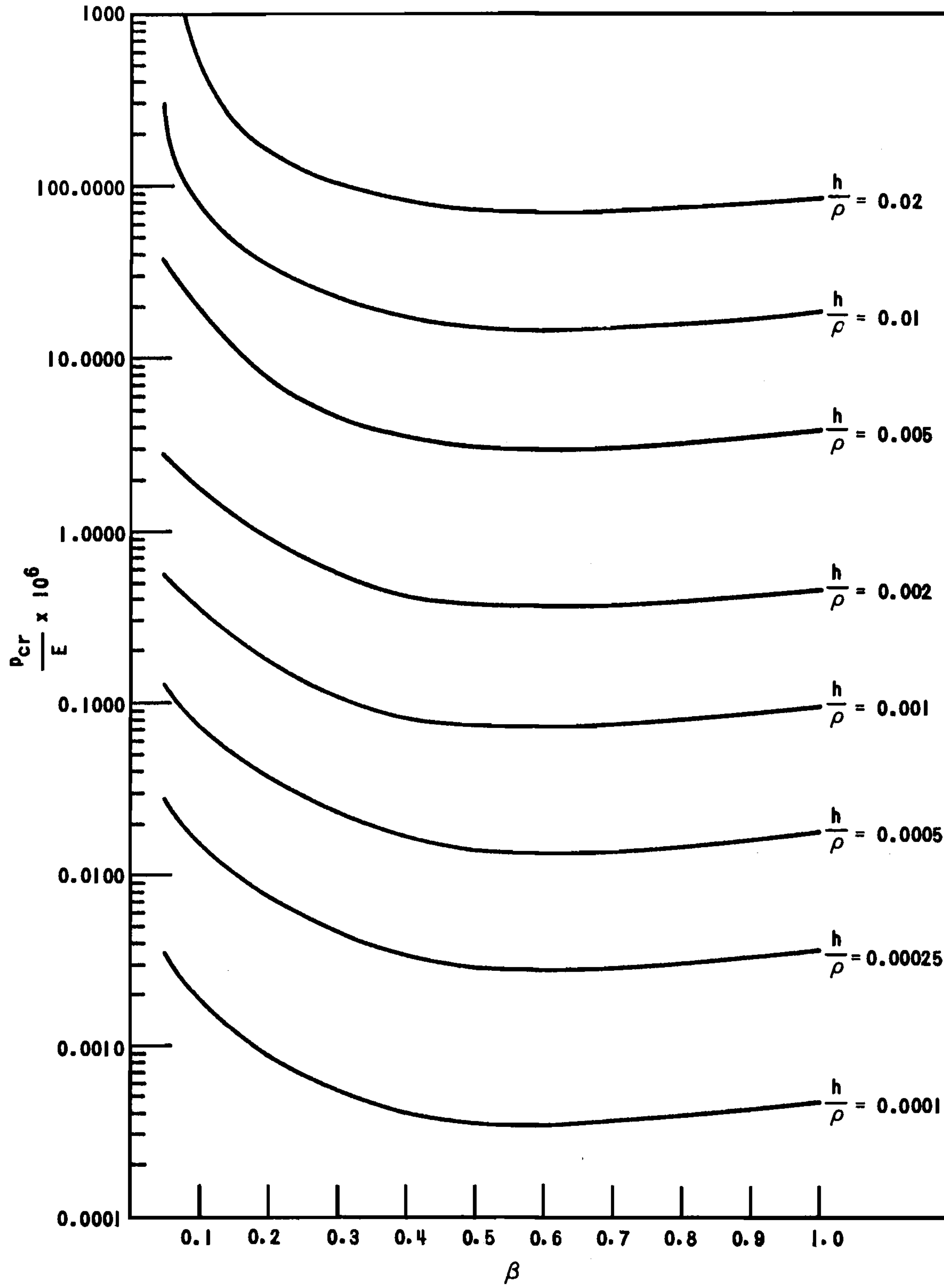
$\alpha = 15^{\circ}$

FIGURE 3



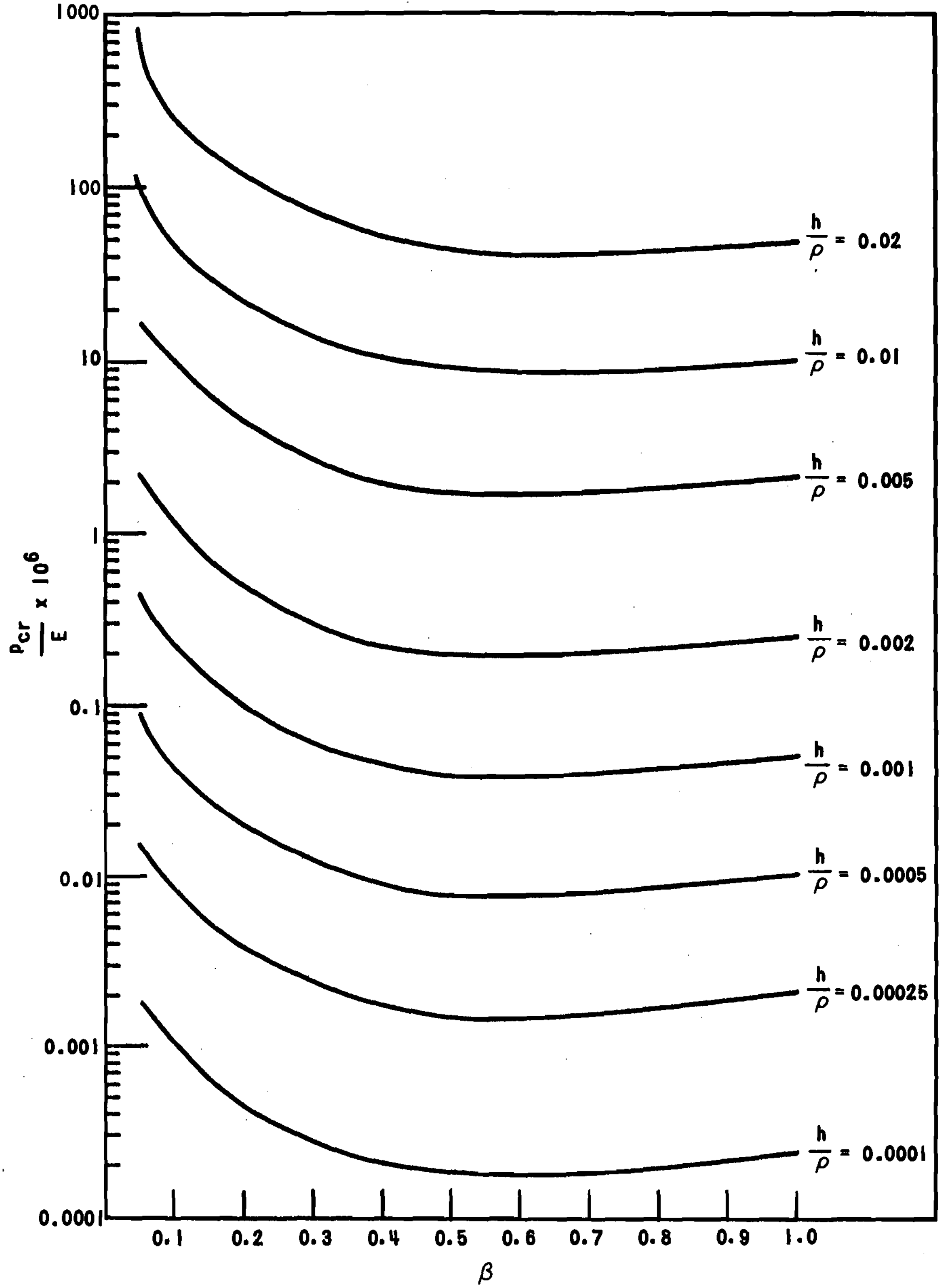
CRITICAL PRESSURE FOR CONICAL FRUSTRA
$\alpha = 30^{\circ}$

FIGURE 4



CRITICAL PRESSURE FOR CONICAL FRUSTRA
$\alpha = 45^{\circ}$

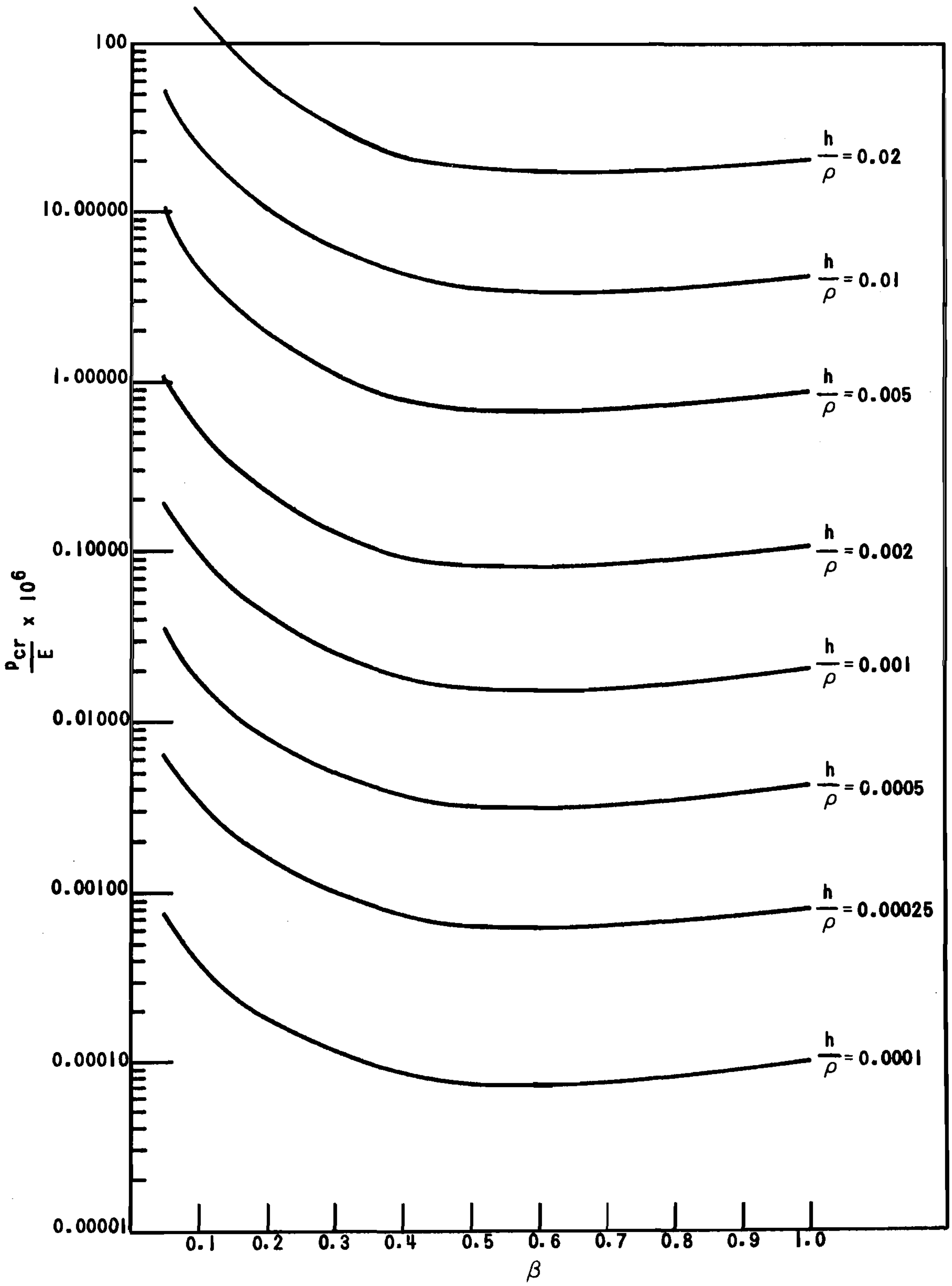
FIGURE 5



CRITICAL PRESSURE FOR CONICAL FRUSTRA

$\alpha = 60^{\circ}$

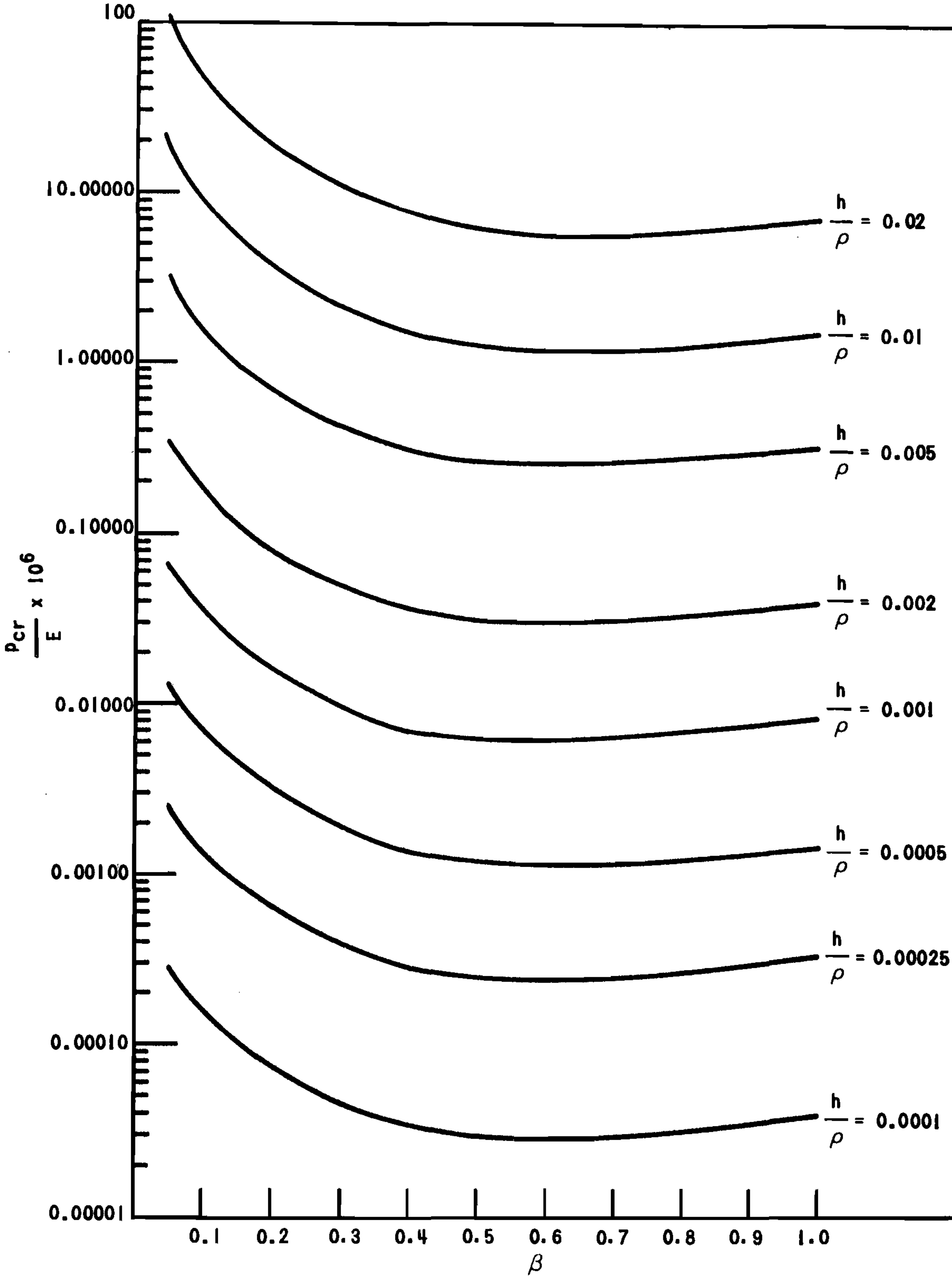
FIGURE 6



CRITICAL PRESSURE FOR CONICAL FRUSTRA

$\alpha = 75^{\circ}$

FIGURE 7



CRITICAL PRESSURE FOR CONICAL FRUSTRA
$\alpha = 85^{\circ}$

FIGURE 8

APPENDIX A

DERIVATION OF STRAIN-DISPLACEMENT RELATIONS

The exact nonlinear strain-displacement relations for curvilinear coordinates have been formulated and are well known. See, for example, Novozhilov[10] page 59, where they are presented in the form:

$$
\begin{aligned}
\epsilon_{11} &= e_{11} + \frac{1}{2}\left[e_{11}^2 + \left(\frac{1}{2}e_{12} + \omega_3\right)^2 + \left(\frac{1}{2}e_{13} - \omega_2\right)^2\right] \\
\epsilon_{22} &= e_{22} + \frac{1}{2}\left[e_{22}^2 + \left(\frac{e_{12}}{2} - \omega_3\right)^2 + \left(\frac{e_{23}}{2} + \omega_1\right)^2\right] \\
\epsilon_{33} &= e_{33} + \frac{1}{2}\left[e_{33}^2 + \left(\frac{e_{13}}{2} + \omega_2\right)^2 + \left(\frac{e_{23}}{2} - \omega_1\right)^2\right] \\
\epsilon_{12} &= e_{12} + e_{11}\left(\frac{e_{12}}{2} - \omega_3\right) + e_{22}\left(\frac{e_{12}}{2} + \omega_3\right) \\
&\quad + \left(\frac{e_{13}}{2} - \omega_2\right)\left(\frac{e_{23}}{2} + \omega_1\right) \\
\epsilon_{13} &= e_{13} + e_{11}\left(\frac{e_{13}}{2} + \omega_2\right) + e_{33}\left(\frac{e_{13}}{2} - \omega_2\right) + \left(\frac{e_{12}}{2} + \omega_3\right)\left(\frac{e_{23}}{2} - \omega_1\right) \\
\epsilon_{23} &= e_{23} + e_{22}\left(\frac{e_{23}}{2} - \omega_1\right) + e_{33}\left(\frac{e_{23}}{2} + \omega_1\right) + \left(\frac{e_{12}}{2} - \omega_3\right)\left(\frac{e_{13}}{2} + \omega_2\right).
\end{aligned}
\qquad \text{(A-1)}
$$

When the usual thin shell orthogonal coordinate system (α_1, α_2, z) is used, (α_1 and α_2 being lines of principal curvature on the shell's middle surface, and z perpendicular to both of these) we have:

$$
\begin{aligned}
e_{11} &= \frac{1}{1 + z/R_1}\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1}\right) \\
e_{22} &= \frac{1}{1 + z/R_2}\left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2}\right) \\
e_{zz} &\equiv e_{zz} = \frac{\partial w}{\partial z}
\end{aligned}
\qquad \text{(A-2)}
$$

$$e_{12} = \frac{1}{(1 + z/R_1)}\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right) + \frac{1}{1 + z/R_2}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)$$

$$e_{13} \equiv e_{1z} = \frac{\partial u}{\partial z} + \frac{1}{1 + z/R_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)$$

$$e_{23} \equiv e_{2z} = \frac{\partial v}{\partial z} + \frac{1}{1 + z/R_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right)$$

$$2\omega_1 = -\frac{\partial v}{\partial z} + \frac{1}{1 + z/R_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right) \qquad \text{(A-2)}$$

$$2\omega_2 = \frac{\partial u}{\partial z} - \frac{1}{1 + z/R_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right)$$

$$2\omega_3 = 2\omega_z = \frac{1}{1 + z/R_1}\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u\right)$$

$$- \frac{1}{1 + z/R_2}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)$$

where: ω_1, ω_2, ω_z are the components of the rotation vector $\bar{\omega}$, R_1R_2 are the principal radii of curvature in the α_1, α_2 directions, u, v, w are the displacements in the α_1, α_2 and z directions, and A_1A_2 are related to the Lamé coefficients of the transformation from rectangular to curvilinear coordinates by:

$$H_1 = A_1\,(1 + z/R_1)\;\; H_2 = A_2\,(1 + z/R_2).$$

By employing the Kirchhoff hypothesis (Reference 10) that "normals to the undeformed middle surface go over into normals to the deformed middle surface, and undergo no extension," and by assuming that the displacements vary linearly in z:

$$u = \hat{u} + z\phi \qquad v = \hat{v} + z\psi \qquad w = \hat{w} + z\chi, \qquad \text{(A-3)}$$

it is possible to express the strain-displacement relations entirely in terms of the middle surface displacements ($\hat{u}$, $\hat{v}$, $\hat{w}$).

Novozhilov (Reference 10) finds, for example, that:

$$\left.\begin{aligned} \phi &= - \hat{e}_{13}(1 + \hat{e}_{22}) + \hat{e}_{23}\,\hat{e}_{12} \\ \psi &= - \hat{e}_{23}(1 + \hat{e}_{11}) + \hat{e}_{13}\,\hat{e}_{21} \\ \chi &= \hat{e}_{11} + \hat{e}_{22} + \hat{e}_{11}\,\hat{e}_{22} - \hat{e}_{12}\,\hat{e}_{21} \end{aligned}\right\} \quad \text{(A-4)}$$

where

$$\left.\begin{aligned} e_{11} &= \frac{1}{A_1}\frac{\partial \hat{u}}{\partial \alpha_1} + \frac{\hat{v}}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\hat{w}}{R_1} \\ e_{22} &= \frac{1}{A_2}\frac{\partial \hat{v}}{\partial \alpha_2} + \frac{\hat{u}}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\hat{w}}{R_2} \\ e_{12} &= \frac{1}{A_1}\frac{\partial \hat{v}}{\partial \alpha_1} - \frac{\hat{u}}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} \\ e_{13} &= \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} - \frac{\hat{u}}{R_1} \\ e_{21} &= \frac{1}{A_2}\frac{\partial \hat{u}}{\partial \alpha_2} - \frac{\hat{v}}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} \\ e_{23} &= \frac{1}{A_2}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{\hat{v}}{R_2} \end{aligned}\right\} \quad \text{(A-5)}$$

Substitution of these relations into Equation A-1 yields what Novozhilov terms a "strong bending theory" - viz., one in which no assumptions have been made on the magnitude of deformation other than that it is one of small strains. This formulation, however, is not only unwieldy - it is unnecessarily general for most thin shell purposes.

Most authors simplify the above equations by using an order of magnitude argument in comparing the various terms involved in the strain expressions. However, this procedure fails to give proper attention to the fact that buckling is a geometrically nonlinear, if physically linear, phenomenon. By physically linear we mean that elongations and shear are much less than unity. (small strains) and do not exceed the proportional limit. This justifies the use of a linear stress-strain law (Hooke's law). By geometrically nonlinear we mean that deformations may be quite large (despite small strains) due to the existence of angles of rotation so large that they may not be neglected in the determination of the strains.

Novozhilov shows that when elongations and shear are small compared to unity and rotations are less than unity but large compared to elongation and shear, then it is necessary to retain only those terms of the order of the squares of the rotations in the nonlinear portion of the strain expressions.

For thin shells $[(1 + z/R_1) \approx 1, (1 + z/R_2) \approx 1]$, and assuming validity of Kirchhoff's hypothesis $(\epsilon_{zz} \approx \epsilon_{1z} \approx \epsilon_{2z} \approx 0)$ we may employ the following approximate set of strain displacement relations. We write

$$\left.\begin{aligned} \epsilon_{11} &= e_{11} + \frac{1}{2}(\omega_2^2 + \omega_3^2) \\ \epsilon_{22} &= e_{22} + \frac{1}{2}(\omega_1^2 + \omega_3^2) \\ \epsilon_{12} &= e_{12} - \omega_1\omega_2 \,. \end{aligned}\right\} \qquad \text{(A-6)}$$

Introducing Equations A-2, A-3 and A-4 into these, we have as the basis for an "intermediate" bending theory of thin shells (analagous in its degree of accuracy to the nonlinear VonKarman plate equations):

$$\left.\begin{aligned} \epsilon_{11} &= \hat{e}_{11} + \frac{1}{2}(\hat{\omega}_2^2 + \hat{\omega}_3^2) + z\left(k_{11} - \frac{\hat{\omega}_2 k_{13}}{2} + \hat{\omega}_3\left[\frac{k_{12} - k_{21}}{2}\right]\right) + z^2\eta_{11} \\ \epsilon_{22} &= \hat{e}_{22} + \frac{1}{2}(\hat{\omega}_1^2 + \hat{\omega}_3^2) + z\left(k_{22} + \frac{\hat{\omega}_1 k_{23}}{2} + \hat{\omega}_3\left[\frac{k_{12} - k_{21}}{2}\right]\right) + z^2\eta_{22} \\ \epsilon_{12} &= \hat{e}_{12} + \hat{e}_{21} - \hat{\omega}_1\hat{\omega}_2 + z\left(k_{12} + k_{21} + \frac{\hat{\omega}_1 k_{13}}{2} - \frac{\hat{\omega}_2 k_{23}}{2}\right) + z^2\eta_{12} \end{aligned}\right\} \qquad \text{(A-7)}$$

where:

$$
\left.
\begin{aligned}
\hat{e}_{11} &= \frac{1}{A_1}\frac{\partial \hat{u}}{\partial \alpha_1} + \frac{\hat{v}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\hat{w}}{R_1} \\
\hat{e}_{22} &= \frac{1}{A_2}\frac{\partial \hat{v}}{\partial \alpha_2} + \frac{\hat{u}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\hat{w}}{R_2} \\
\hat{e}_{12} &= \frac{1}{A_1}\frac{\partial \hat{v}}{\partial \alpha_1} - \frac{\hat{u}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \\
\hat{e}_{21} &= \frac{1}{A_2}\frac{\partial \hat{u}}{\partial \alpha_2} - \frac{\hat{v}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \\
\hat{e}_{13} &= \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} - \frac{\hat{u}}{R_1} \\
\hat{e}_{23} &= \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{\hat{v}}{R_2} \\
2\hat{\omega}_1 &= -\psi + \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_2} - \frac{\hat{v}}{R_2} \\
2\hat{\omega}_2 &= \phi - \frac{1}{A_1}\frac{\partial \hat{w}}{\partial \alpha_1} + \frac{\hat{u}}{R_1} \\
2\hat{\omega}_3 &= \frac{1}{A_1}\frac{\partial \hat{v}}{\partial \alpha_1} - \frac{\hat{u}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{1}{A_2}\frac{\partial \hat{u}}{\partial \alpha_2} + \frac{\hat{v}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \\
k_{11} &= \frac{1}{A_1}\frac{\partial \phi}{\partial \alpha_1} + \frac{\psi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\chi}{R_1} \\
k_{12} &= \frac{1}{A_1}\frac{\partial \psi}{\partial \alpha_1} - \frac{\phi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \\
k_{13} &= \frac{1}{A_1}\frac{\partial \chi}{\partial \alpha_1} - \frac{\phi}{R_1}
\end{aligned}
\right| \qquad \text{(A-8)}
$$

$$k_{22} = \frac{1}{A_2}\frac{\partial\psi}{\partial\alpha_2} + \frac{\phi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \chi/R_2$$

$$k_{21} = \frac{1}{A_2}\frac{\partial\phi}{\partial\alpha_2} - \frac{\psi}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \qquad \text{(A-8)}$$

$$k_{23} = \frac{1}{A_2}\frac{\partial\chi}{\partial\alpha_2} - \psi/R_2$$

and, finally:

$$\left.\begin{aligned} \phi &= -\hat{e}_{13}(1+\hat{e}_{22}) + \hat{e}_{23}\hat{e}_{12} \approx -\hat{e}_{13} \\ \psi &= -\hat{e}_{23}(1+\hat{e}_{11}) + \hat{e}_{13}\hat{e}_{21} \approx -\hat{e}_{23} \\ \chi &= \hat{e}_{11} + \hat{e}_{22} + \hat{e}_{11}\hat{e}_{22} - \hat{e}_{12}\hat{e}_{21} \approx \hat{e}_{11} + \hat{e}_{22}. \end{aligned}\right\} \qquad \text{(A-9)}$$

We observe that to be consistent we neglect terms of the order ${e_{ij}}^2$ in ϕ, ψ and χ since we have already neglected terms of this order in ϵ_{ij}. We also neglect terms of the order $z^2\eta_{ij}$.

APPENDIX B

DETERMINATION OF CHANGE IN POTENTIAL OF EXTERNAL FORCES

For the undeformed cone, a point on the middle surface of the shell is given by the coordinates (ξ, θ). These may be related to rectangular coordinates by means of the equations:

$$x = r\cos\theta = (ro - \xi\cos\alpha)\cos\theta$$
$$y = r\sin\theta = (ro - \xi\cos\alpha)\sin\theta$$
$$z = \xi\sin\alpha \qquad \text{(B-1)}$$

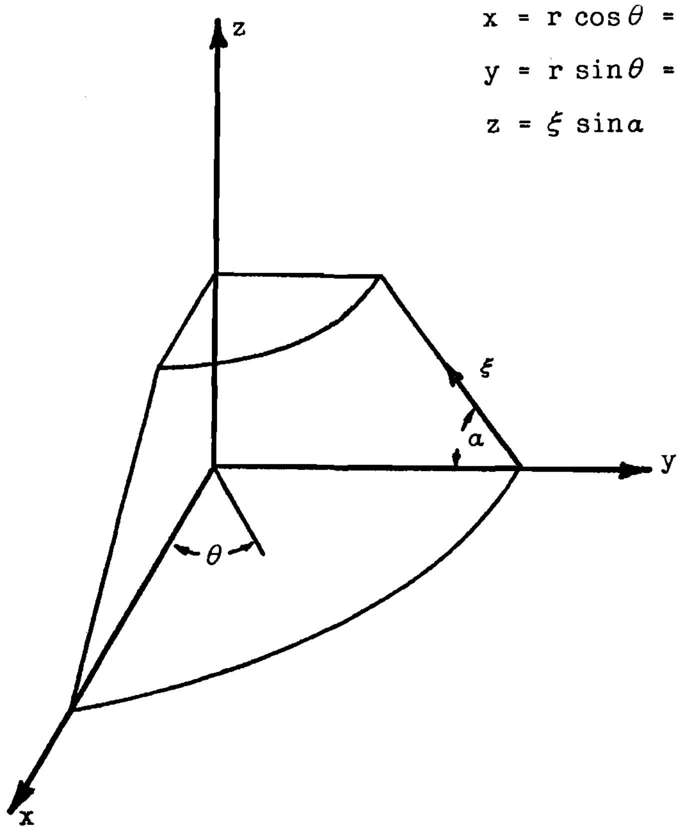


FIGURE (B-1)

For the deformed cone, a point having original coordinates ξ, θ has final rectangular coordinates X, Y, Z given by:

$$\left.\begin{aligned} X &= (r + w)\cos\theta - v\sin\theta \\ Y &= (r + w)\sin\theta + v\cos\theta \\ Z &= (\xi + u)\sin\alpha . \end{aligned}\right\} \qquad \text{(B-2)}$$

From these we obtain Gauss' fundamental coefficients E, F and G by means of the relations:

$$
\left.
\begin{aligned}
E &= X_\xi^2 + Y_\xi^2 + Z_\xi^2 \\
&= \{(1 + u_\xi)^2 \sin^2\alpha + (w_\xi - \cos\alpha)^2 + v_\xi^2\} \\
F &= X_\xi X_\theta + Y_\xi Y_\theta + Z_\xi Z_\theta \\
&= (1 + u_\xi) u_\theta \sin^2\alpha + v_\xi (v_\theta + w + r) + (w_\theta - v)(w_\xi - \cos\alpha) \\
G &= X_\theta^2 + Y_\theta^2 + Z_\theta^2 \\
&= (u_\theta \sin\alpha)^2 + (r + w + v_\theta)^2 + (w_\theta - v)^2.
\end{aligned}
\right| \qquad \text{(B-3)}
$$

An element of area on the (deformed) curved surface is given by

$$
dS = \sqrt{EG - F^2}\; d\xi d\theta \equiv D d\xi d\theta, \qquad \text{(B-4)}
$$

hence the volume enclosed by the deformed middle surface is:

$$
\iiint dv = \iint Z n_z dS = \iint_{\text{lat. surface}} Z n_z D d\xi d\theta + \iint_{\text{ends}} Z dS \qquad \text{(B-5)}
$$

where n_z is the z component of the unit normal vector to the deformed middle surface.

But from differential geometry (Reference 3),

$$
\left.
\begin{aligned}
D n_z &= X_\theta Y_\xi - Y_\theta X_\xi \\
&= v_\xi (w_\theta - v) - (r + w)_\xi (v_\theta + r + w).
\end{aligned}
\right\} \qquad \text{(B-6)}
$$

Now the volume enclosed by the undeformed conical shell is given by:

$$
V_1 = \frac{l \sin\alpha}{3} \left(A_1 + A_2 + \sqrt{A_1 A_2} \right) \qquad \text{(B-7)}
$$

where A_1 and A_2 are the area of the top and bottom cross section of the cone.

Hence, the change in volume during deformation is:

$$\left.\begin{aligned} \Delta V = & \iint \sin\alpha\,(\xi + u)\{v_\xi(w_\theta - v) - (r + w)_\xi(v_\theta + w + r)\}d\xi d\theta \\ & + \sin\alpha\left(lA_1 + e(A_1A_2)\right) - \frac{l\sin\alpha}{3}\left(A_1 + A_2 + \sqrt{A_1 + A_2}\right) \end{aligned}\right\} \quad \text{(B-8)}$$

where e is the average extension of the ends of the cone. The work done by the uniform external pressure is then given by $p \cdot \Delta V$.

APPENDIX C

(a) TABULATION OF I* THROUGH XIX**

$$I^* = \beta^4(2.19846) - \beta^3(10.2868) + \beta^2(20.1370) - \beta(21.6493) + 12.3367 - \beta^{-1}(4.93479)$$

$$II^* = \beta^2(4.60870) - \beta(11.6848) + 7.40219 - \beta^{-1}(4.93479)$$

$$III^* = \beta^4(-.161731) + \beta^3(.709972) + \beta^2(-1.18121) + \beta(.942477) + (-.471239)$$

$$IV^* = \beta^2(-.682751) + \beta(1.570795) + (-1.570795)$$

$$V^* = \beta^5(-1.159947) + \beta^4(+6.903993) + \beta^3(-20.51448) + \beta^2(30.16964) + \beta(-24.7400) + (9.89600)$$

$$VI^* = \beta^2(4.801058) + \beta(-16.82460) + (21.47553) + \left(\frac{1}{\beta}\right)(-9.30186) + \left(\frac{1}{\beta^2}\right)(4.65093)$$

$$VII^* = \beta^4(.087005) + \beta^3(-.424009) + \beta^2(.75) + \beta(-.5)$$

$$VIII^* = \beta^5(-.250022) + \beta^4(1.50159) + \beta^3(-3.62159) + \beta^2(4.24) + \beta(-2.12)$$

$$IX^* = \beta^5(.0285192) + \beta^4(-.174009) + \beta^3(.424008) + \beta^2(-.5) + \beta(.25)$$

$$X^* = \beta^4(.910570) + \beta^3(-5.3610) + \beta^2(13.08704) + \beta(-16.9493) + 12.33699 + (1/\beta)(-4.93479)$$

$$XI^* = \beta^2(36.8676) + \beta(-162.955) + (292.0113) + (1/\beta)(-275.848)$$
$$+ (1/\beta^2)(121.761) + (1/\beta^3)(-48.7044)$$

$$XII^* = \beta^2(1.608703) + \beta(-5.68480) + (7.40219) + (1/\beta)(-4.93479)$$

$$XIII^* = \beta^2(.25) + (\beta)(-.5)$$

$$XIV^* = \beta^2(-2.059795) + \beta(8.314152) + (-12.38122) + (1/\beta)(8.38915)$$

$$XV^* = \beta^5(-1.79703) + \beta^4(12.49638) + \beta^3(-37.3091) + \beta^2(61.96213)$$
$$+ \beta(-61.82351) + (37.01096) + (1/\beta)(-12.33699)$$

$$XVI^* = \beta^5(-.625055) + \beta^4(3.75397) + \beta^3(-9.053968) + \beta^2(10.60)$$
$$+ \beta(-5.30)$$

$$XVII^* = \beta^2(-.157609) + \beta(1.05544) + (-2.220657) + (1/\beta)(1.48044)$$

$$XVIII^* = \beta^2(4.93479) + \beta(-19.73919) + (29.60876) + (1/\beta)(-19.73917)$$

$$XIX^* = \beta^5(-.171116) + \beta^4(1.044054) + \beta^3(-2.54405) + \beta^2(3.0)$$
$$+\beta(-1.5).$$

(b) TABULATION OF I_1^* THROUGH I_{11}^{**}

$$I_1^{**} = \int_0^\pi \frac{\sin^2 x\, dx}{1 - ax} = -\frac{d}{2}\left[\log(1 - a\pi) - \left(\cos 2d\, Ci\,(Y) + \sin 2d\, Si\,(Y)\right)\Big|_{2d}^{2b}\right]$$

$$I_2^{**} = \int_0^\pi \frac{\cos^2 x\, dx}{1 - ax} = -d\log(1 - a\pi) - I_1^{**}$$

$$I_3^{**} = \int_0^\pi \frac{\sin 2x\, dx}{1 - ax} = -d\ [\sin 2d\, Ci\,(Y) - \cos 2d\, Si\,(Y)]_{2d}^{2b}$$

$$I_4^{**} = \int_0^\pi \frac{\cos^2 x\, dx}{(1 - ax)^3} = \frac{d}{2}\left[\frac{1}{(1 - a\pi)^2} - 1\right] - I_6^{**}$$

$$I_5^{**} = \int_0^{\pi} \frac{\sin 2x \, dx}{(1 - ax)^3} = d^3 \left[\frac{1}{d} - \frac{1}{b} \right] - 2d^2 I_3^{**}$$

$$I_6^{**} = \int_0^{\pi} \frac{\sin^2 x \, dx}{(1 - ax)^3} = - d^3 \log(1 - \beta) - 2d^2 I_1^{**}$$

$$I_7^{**} = \int_0^{\pi} \frac{\sin 2x \, dx}{(1 - ax)^2} = 2d^2 \log(1 - a\pi) + 4d I_1^{**}$$

$$I_8^{**} = \int_0^{\pi} \frac{\sin^2 x \, dx}{(1 - ax)^2} = - dI_3^{**}$$

$$I_9^{**} = \int_0^{\pi} \frac{\cos^2 x \, dx}{(1 - ax)^2} = d\left(I_3^{**} + \frac{\pi}{b} \right)$$

$$I_{10}^{**} = \int_0^{\pi} \frac{x \sin^2 x \, dx}{(1 - ax)^2} = - d I_1^{**} - d^2 I_3^{**}$$

$$I_{11}^{**} = \int_0^{\pi} \frac{x^2 \sin^2 x \, dx}{(1 - ax)^2} = - 2d^2 I_1^{**} - d^3 I_3^{**} + \frac{\pi}{2} d^2$$

where:

$$d \equiv \frac{1}{a} = \frac{\pi}{\beta} \qquad b \equiv d - \pi$$

and

$$Ci(Y) \Big|_{2d}^{2b} \equiv \int_{2d}^{2b} \frac{\cos y}{y} \, dy$$

$$Si(Y) \Big|_{2d}^{2b} \equiv \int_{2d}^{2b} \frac{\sin y}{y} \, dy \, .$$

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